DIRECT STOCHASTIC TREATMENT OF ELECTROMAGNETIC PROPAGATION PROBLEMS IN A RANDOM RANGE-DEPENDENT MEDIUM, USING A PE MODEL

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1. Introduction

The atmospheric turbulence is known to have significant effects on radiowave propagation ; they are mainly characterized by scintillations in the wavefront. It is particularly true for frequencies over 10 GHz, inside the boundary layer where the turbulence intensity is stronger than in free atmosphere, and in the presence of a duct, at observation points where destructive interference from multipaths can yield a null in the coherent field.

A standard way to describe the refractive index fluctuations due to turbulent eddies consists in adding to the mean refractive profile a random component n_1, assumed to depend upon the sole vertical coordinate :

$$n(r,s) = \langle n_0(r) \rangle + n_1(r,s) \text{ , s: range, r: altitude} \qquad (1)$$

Since the turbulent motions of the atmospheric cells are highly chaotic, and eddy size can range from 1 mm to one hundred meters, a deterministic description of n_1 is not possible. So we classically characterize the turbulent medium through its statistical properties, for instance the correlation function or the density power spectrum. The electromagnetic field in a turbulent medium is consequently known by its statistics too : mainly, first and second order moment or mean electromagnetic field and mean intensity.

In this work, we present a direct computation of the statistics of the electromagnetic field in presence of turbulent refractive fluctuations, using a standard parabolic equation (PE) model. This new approach is based on stochastic calculus and more precisely on the Itô formalism which will allow us to generate all the response statistics equations. In addition, the last paragraph will show how to characterize the turbulent medium in connexion with this study and examine some results.

2. **Study framework**

The starting point of this approach is a standard P.E. model, CORAFIN, developed at DCN, associated with a finite difference method using a Crank-Nicolson scheme. The propagation equation takes the parabolic form:

$$\frac{\partial U}{\partial s} \cdot -\frac{j}{2k_0}\frac{\partial^2 U}{\partial r^2} - j\frac{k_0}{2}\left(m^2-1\right)U \tag{2-a}$$

with associated boundary conditions :
$$\begin{cases} \dfrac{\partial U}{\partial r} - jk_0 q_{H,V}\, U = 0 \\ \lim\limits_{r\to\infty}\left(\dfrac{\partial U}{\partial r} + jk_0\, U\right) = 0 \end{cases} \tag{2-b}$$

m is the modified refractive index and $q_{H,V}$ depends upon the polarisation.

Introduction of the random componant n_1, under the form : $n^2 = \langle n_0\rangle^2 + 2n_1$ leads to a Stochastic Partial Differential Equation (SPDE) :

$$\frac{\partial U}{\partial s} - \frac{j}{2\,k_0}\frac{\partial^2 U}{\partial r^2} - j\frac{k_0}{2}\left(m^2-1\right)U - jk_0 n_1\, U \tag{3}$$

in which the last coefficient is a physical random process. The solutions of this equation will be determined by computing their statistic moments.

A classical way to solve this kind of equation and obtain some of the moments is to use a statistical approach like a Monte Carlo method, which means first generating numerical random fluctuations n_1, generally from the turbulence spectrum, then solving the deterministic parabolic equation for each realization of n_z, and last averaging on the whole set of independent simulations.

This method has become very popular, due to increasing computer power. The first order moment is easily generated, as is a particular value of the second order moment : its value at origine. A wide choice of turbulence spectra allows an accurate description of the physical phenomenum. But on the other side, it requires a huge computational time, because the error on the result behaves like the inverse of the square root of the number of simulations. Moreover, it is quite difficult to determine rigorous convergence criterions : the choice of a "right" simulation number, for which the method converges, remains empirical, and in all cases, this number seems to be greater than one hundred trials.

The interest of a direct stochastic treatment of the SPDE is then obvious : if we can derive from the stochastic equation deterministic partial differential equations for all the moments, we shall then obtain each response statistics by solving single equation. The treatment of stochastic differential equations has been discuted extensively in the literature. The main difficulty is a closure problem : "
for instance, getting the first order moment, implies averaging equation (3) :

$$\frac{a(u)}{\partial s} = -\frac{j}{2\,k_0}\frac{\partial^2\langle U\rangle}{\partial r^2}\, j\frac{k_0}{2}\left(m^2 - 1\right)\langle U\rangle - jk_0\langle n_1\, U\rangle \tag{4}$$

(4) contains the first moment $\langle U\rangle$ as expected, but also the crossed moment $\langle n_1\, U\rangle$, for which no statistics are known. To deal with this problem, we thus need closure assumptions ; they will

be obtained from hypothesis on the turbulent process. We propose here an approach based on the stochastic calculus rules. This formalism permits us to derive the first and second order moment equations, but also the higher order moment equations.

3. Stochastic treatment of the SPDE

Using the properties of some random processes, and particularly the Brownian motion, mathematicians have established a number of theorems which define the rules of stochastic calculus. As we will see, these rules slightly differ from ordinary calculus operations. It is particularly the case for the integration operations we will have to treat. But before introducing the stochastic equation formalism, we prefer to transform the SPDE into a stochastic ordinary differential equation. This is easily achieved by semi-discretization in r, and makes things easier. We thus use :

$$\left.\frac{\partial^2 U(r,s)}{\partial r^2}\right|_{r=ih} = \frac{1}{h^2}\left(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)\right) + \mathrm{O}\left(h^2\right) \; i=0,\ldots,\mathrm{M} \tag{5}$$

where h *is* the vertical step, to obtain the ordinary stochastic differential system:

$$\frac{dU_i(s)}{ds} = bU_{i+1}(s) \cdot d_i U_i(s) \cdot bU_{i-1}(s) \cdot \sigma_i(s) U_i(s) \tag{6}$$

where $U_i(s)$ is an $\mathrm{O}(h^2)$-approximation of $U(ih,s)$, $i=O,\ldots,M$

$$\begin{cases} b = -j/2k_0h^2 \\ d_i = -2b - j\frac{k_0}{2}\left(m_i^2 - 1\right), \quad i = 1,\cdots,M-1 \\ \sigma_i = -jk_0 n_1(ih), \qquad i = 0,\cdots,M \end{cases}$$

$$u_{-1} = U_{M+1} = 0$$

$$do = -2b - j\frac{k_0}{2}\left(m_0^2 - 1\right) - 2j\,bhk_0 q_{H,P}$$

$$d_M = -2b - j\frac{k_0}{2}\left(m_M^2 - 1\right) - 2j\,bhk_0$$

This set of M equations can be written under the following matricial form:

$$\frac{d}{ds}\underline{U}(s) = \underline{\underline{f}}\left(\underline{U}(s)\right) + \underline{\underline{G}}\left(\underline{U}(s)\right).\underline{\sigma}(s) \tag{7}$$

where the M-vectors U, σ, and MxM-matrixes f, G are defined by :

$$\underline{U} = \left(U_0,\cdots,U_M\right)^T, \qquad \underline{\sigma} = \left(\sigma_0,\cdots,\sigma_M\right)^T, \qquad \underline{\underline{f}}\left(\underline{U}(s)\right) = \underline{\underline{f}}.\underline{U}(s)$$

$$\underline{\underline{f}} = \begin{bmatrix} d_0 & 2b & 0 & \cdots & 0 \\ & \ddots & & & \vdots \\ & b & d_i & b & \\ & & & \ddots & \\ 0 & \cdots & 0 & 2b & d_M \end{bmatrix}, \qquad \underline{\underline{G}} = \mathrm{diag}(\underline{U}) = \begin{vmatrix} U_0 & 0 & \cdots & & 0 \\ \vdots & \ddots & & & \\ & 0 & U_i & 0 & \\ & & & & \\ 0 & \cdots & & 0 & U_M \end{vmatrix}$$

At this point, we have transformed the initial SPDE under a more suitable form. Matricial system (7) describes the response U of a structural system under a wide band random parametric excitation σ. To apply standard results on stochastic differential equation, the main assumption is now that the physical random process σ is close to a gaussian white noise, defined as the formal derivative of a Brownian motion B :

$$g(s) \approx \frac{d\underline{B}(s)}{ds} \tag{8}$$

Going back to the physics, the underlying assumptions on the turbulent process are then :

(HI) the random process is Gaussian, i.e.completely described by its first and second moments,

(H2) the random fluctuations are zero-mean $\langle n_1(r,s)\rangle = 0$

(H3) the random medium is delta-correlated in the propagation direction. The correlation function takes the form: $\langle n_1(r,s)\, n_1(r', s')\rangle = \delta(s - s')A(r,r')$

Two additional hypothesis are made for the sake of simplicity :

(H4) the medium is homogeneous, with correlation function :

$A(r,r') = A(r - r', (r + r')/2)$, where the second argument is the mean height.

(H5) the medium is isotropic :

$A(r,r') = A(|r - r'|,(r + r')/2)$

Under *(HI)* to *(H5)*, σ can be replaced by dB and system (7) becomes :

$$d\underline{U}(s) = \underline{\underline{f}}(\underline{U}(s))ds + \underline{\underline{G}}(\underline{U}(s)).d\underline{B}(s) \tag{9}$$

By integration, we obtain :

$$\underline{U}(s) - \underline{U}(0) = \int_0^s \underline{\underline{f}}(\underline{U}(s))ds + \int_0^s \underline{\underline{G}}(\underline{U}(s)).d\underline{B}(s) \tag{1o}$$

In the right handside of (10), the first term is an ordinary Riemann integral because the matrix f has "deterministic" elements. But the last integral is a stochastic integral and can not be treated with ordinary rules, since the brownian motion process B is continuous with unbounded variations. The main problem of this approach is thus the definition of this stochastic integral.

For physical reasons, we chose the Stratonovich stochastic integral, and obtain the following final form of (9):

$$d\underline{U}(s) = \underline{\underline{F}}(\underline{U}(s))ds + \underline{\underline{G}}(\underline{U}(s))\cdot d\underline{B}(s)\,, \qquad F_i(\underline{U}(s)) = f_i(\underline{U}(s)) + \frac{1}{2}k_0^2 A_{ii}\, U_i(s) \tag{11}$$

We have thus obtained a general stochastic equation for the full stochastic field. From this equation, we can derive deterministic (it means ordinary) partial differential equations for all the moments of the stochastic process U. Numerical treatments can then be applied.

4. First and second order moments

It is now easy to derive the partial differential equations for the first and second moments.

We obtain the following ordinary partial equation for the first moment :

$$\frac{d}{ds}E[U_i] = E[\{F_i(\underline{U})\}] = E[f_i(\underline{U})] - \frac{k_0^2}{2}A(0,ih)E[U_i] \tag{12}$$

which appears to be the semi-discretized form of:

$$\frac{\partial}{\partial s}E[U(r,s)] = \left(-\frac{j}{2k_0}\frac{\partial^2}{\partial r^2} - j\frac{k_0}{2}\left(m^2 - 1\right) - \frac{k_0^2}{2}A(0,r)\right)E[U(r,s)] \tag{13}$$

It is very interesting to note that in this case of the first order moment, we still obtain a parabolic equation. The numerical implementation thus implies only a slight change in the existing code.

In the same way, to obtain for the second moment the following equation :

$$\frac{\partial}{\partial s}E[U(r_k,s)U(r_l,s)] = \left[-\frac{j}{2k_0}\left(\frac{\partial^2}{\partial r_k^2} - \frac{\partial^2}{\partial r_l^2}\right) - j\frac{k_0}{2}\left(m^2(r_k,s) - m^2(r_l,s)\right)\right.$$
$$\left. - \frac{k_0^2}{2}\left(A\left(0,\frac{r_k+r_l}{2}\right) - A\left(|r_k - r_l|,\frac{r_k+r_l}{2}\right)\right)\right]E[U(r_k,s)U(r_l,s)] \tag{14}$$

This higher order moment equation is of course more complicated and its numerical treatment is specific.

Remark : Our results are a generalisation of those obtained by Tatarski using an other formalism (Novikov-Furutsu formulae) [4]. In our case, the assumptions on the medium are less restrictive, giving the possibility to take into account a non homogeneous turbulence in altitude, through the expression of the correlation function A.

5. First order moment coverage : results

According to paragraph 4, we can compute the mean loss coverage of the field in a turbulent medium by a slight change of the existing parabolic code. A simple validation of this method consists then in comparing its numerical results with those obtained by a statistical approach (Monte-Carlo method), using a random profile generator [2][3].

Notice that the cost of a Monte Carlo method is approximatively a hundred times (in fact, the number of trials times) the cost of our direct method.

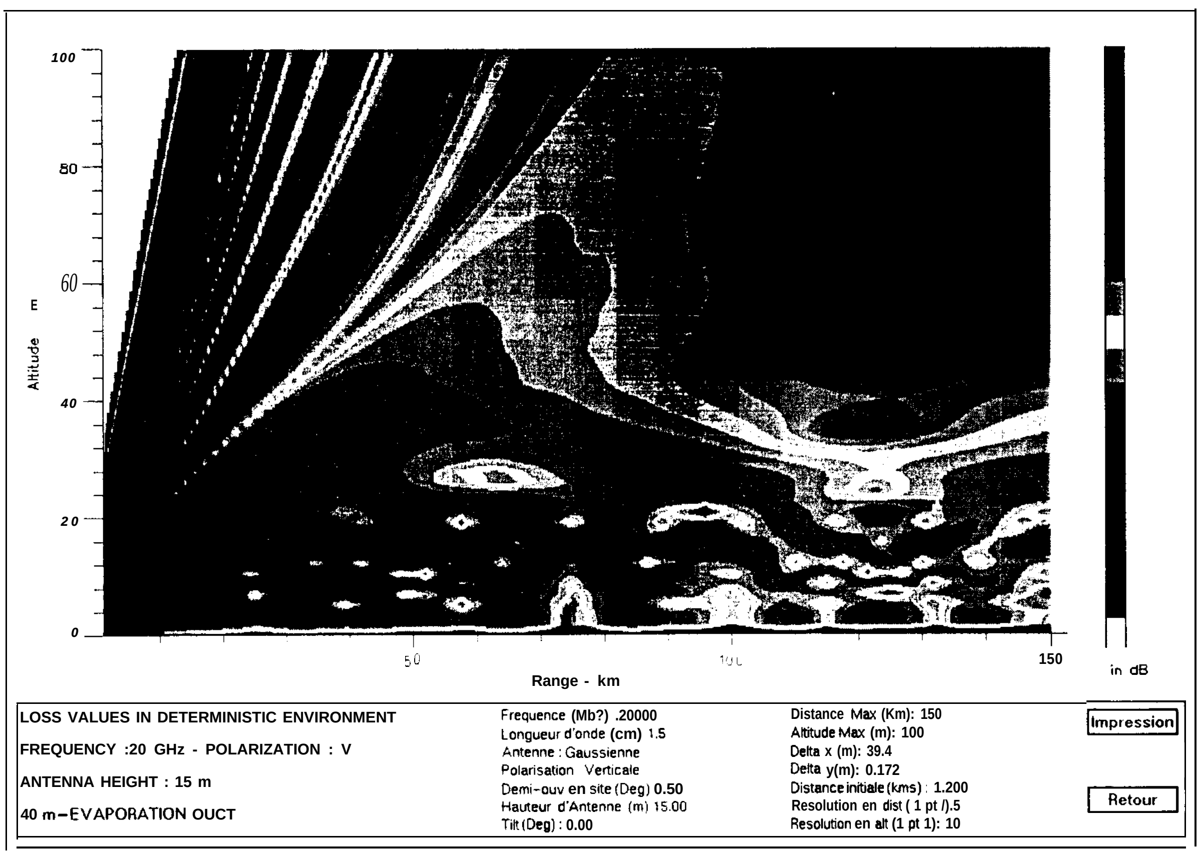
In the following example, the turbulence is described by a Kolmogorov model. As in Kolmogorov theory no correlation function is defined, the difficulty was to derive from the structure function the correlation function term which appears in our equation. This was done using relationship [4] :

$$A(0,r) = \langle n_1^2 \rangle = \frac{C_n^2(r)}{1.91\, L_0^{-2/3}}$$

where the constant structure C_n^2 is given by the Gossard model, well suited to the evaporation duct case, *Lo* being the external length of the turbulence.

A radar frequency of 20 GHz frequency propagates, in presence of a strong 40m evaporation duct (Battaglia model). The electromagnetic loss coverage is computed up to 100 meters versus 150 kilometers range. The antenna height is 15 meters. Figure 1 below shows the “deterministic” radar coverage.

fig. 1



Figures 2-a,b present the difference in dB beetwen the results in deterministic medium and in turbulent medium : 2-a for the stochastic approach, 2-b for the statistical approach using 120 simulations. In both cases, this difference can reach 14 dB, but this strong value appears at points where we observe destructive interference from multipaths. We can see the very good agreement between the two methods. We remind that the second case is 120 times more expansive than the first one.

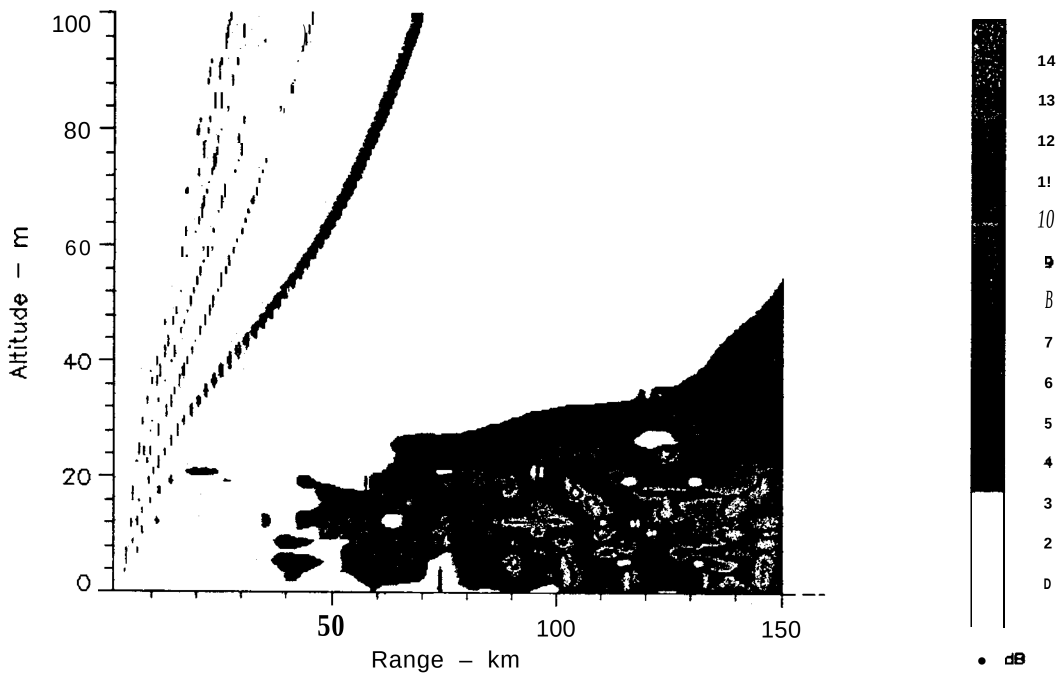


Fig 2-a

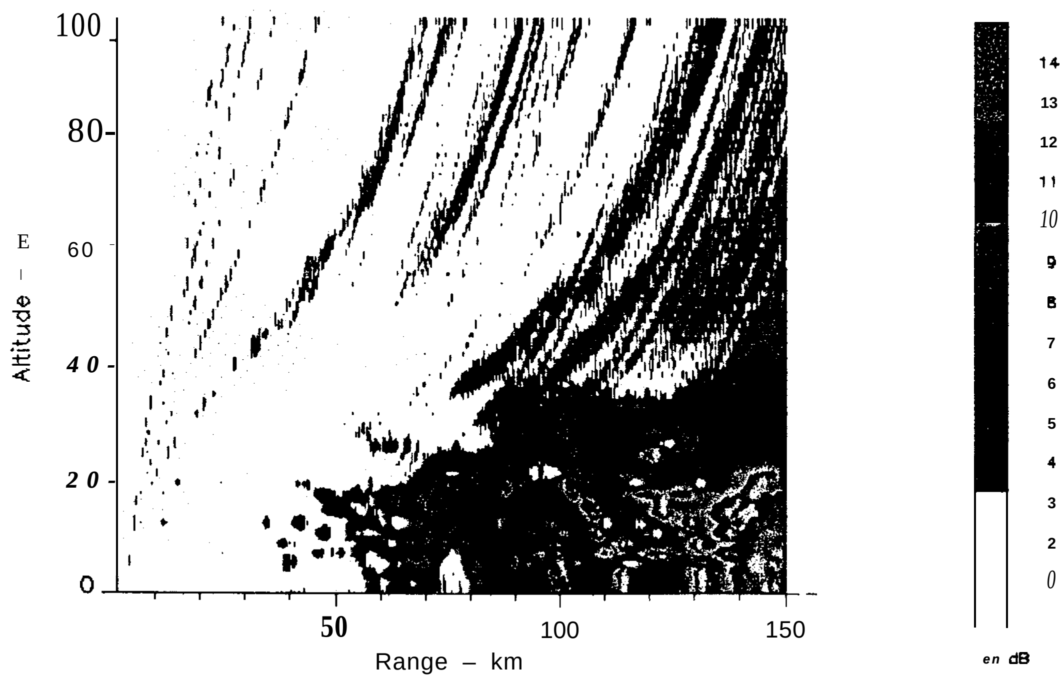


Fig 2-b

The same agreement can also be observed on the following range loss diagrams at altitudes 10 m. (fig. 3), 20m. (fig. 4) and 60 m. (fig. 5). Each fig. (-a) contains the "non-turbulent" result (continuous curve) followed by the stochastic and statistical turbulent results (dashed curves). The second boxes (-b) give the difference beetwen turbulent and non-turbulent cases.

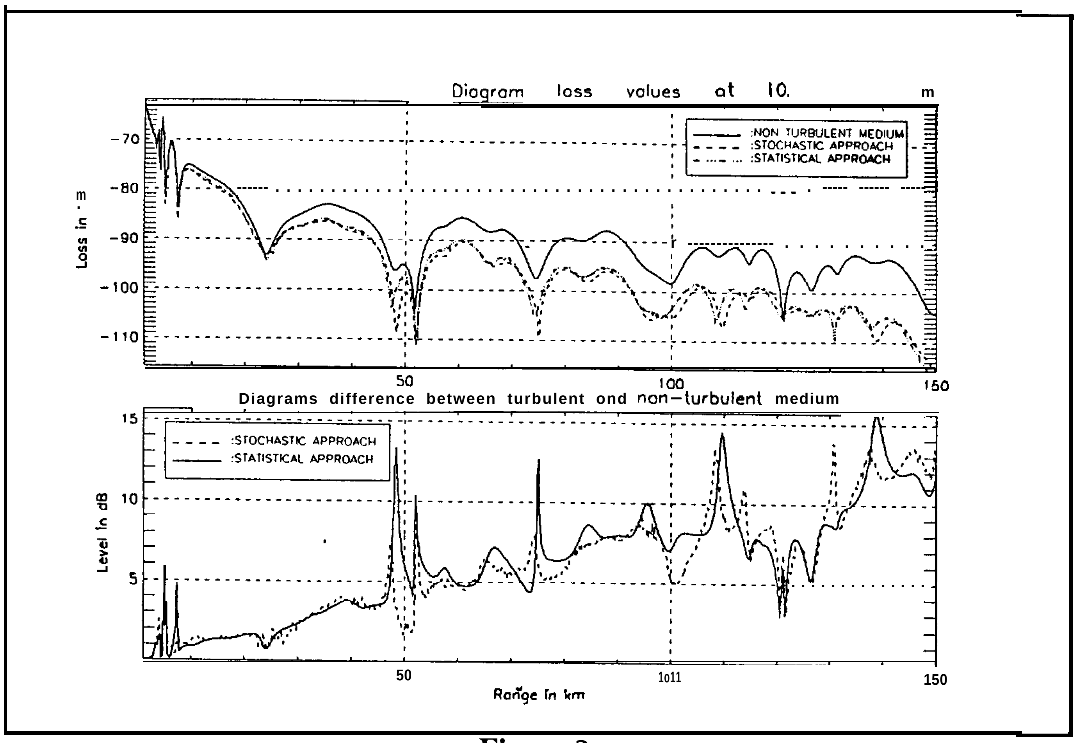


Figure 3

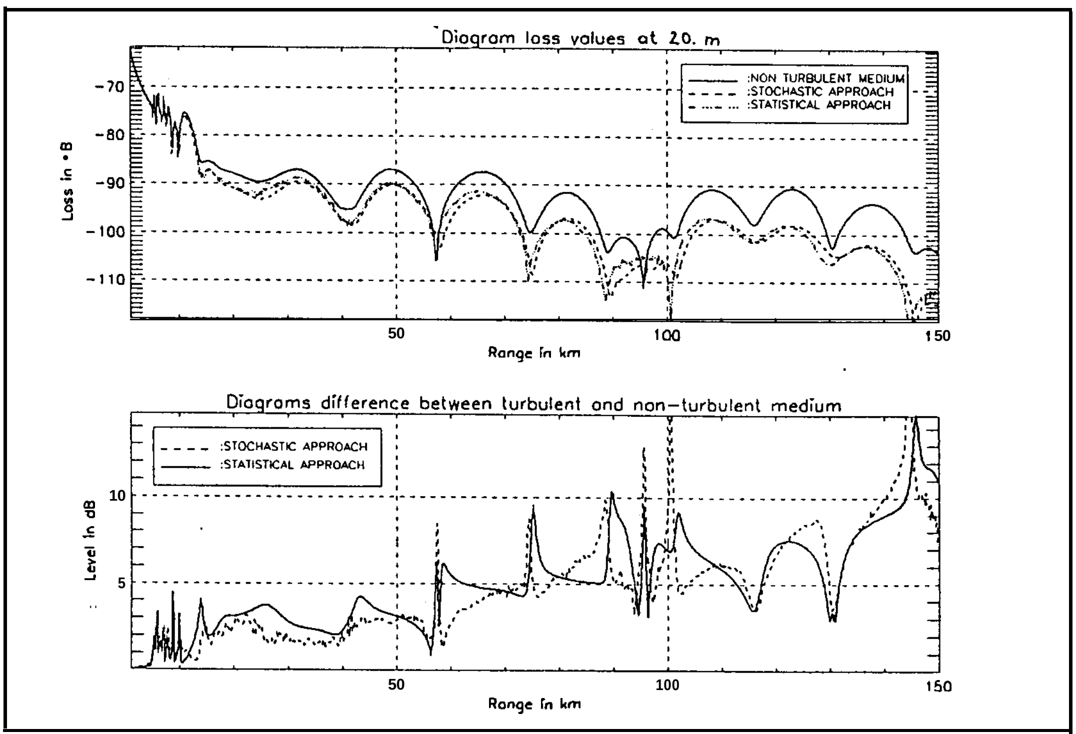


Figure 4

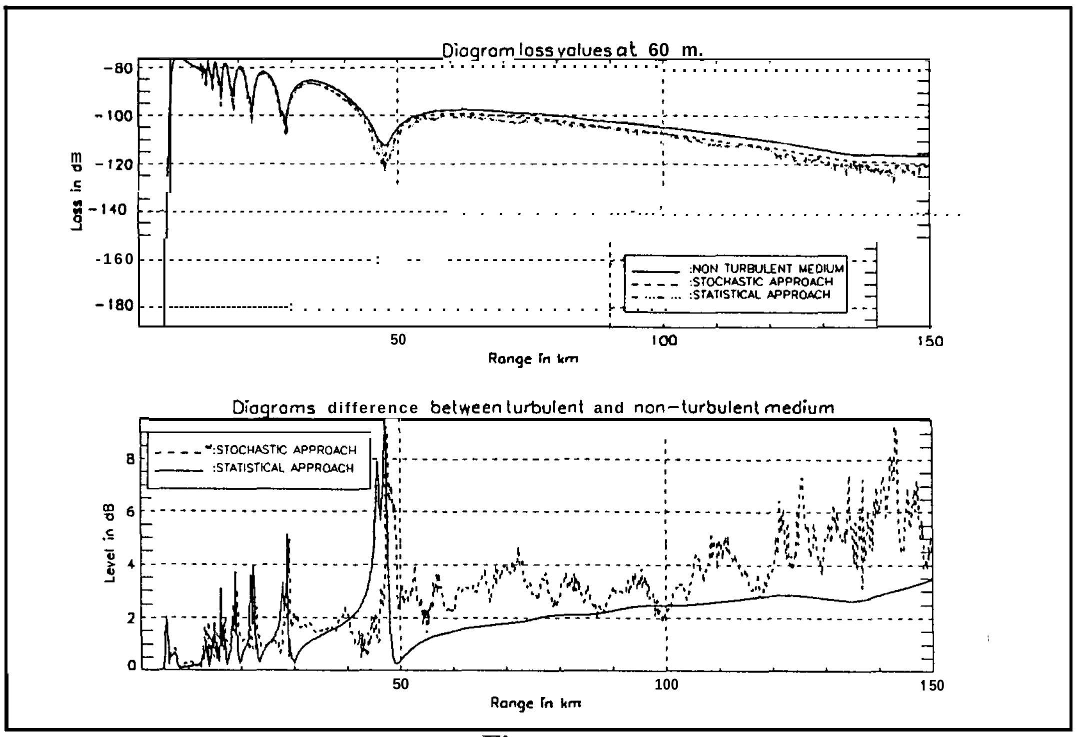


Figure 5

6. Conclusion

We propose a direct stochastic treatment to solve the problem of radar wave propagation in a random turbulent medium. The Stochastic Differential Equations formalism allows us to compute the mean electromagnetic field (first moment) coverage in a turbulent medium by solving only ONE parabolic equation, while a statistical approach needs more than 100 independent simulations. The numerical results are shown to be in very good agreement for the two methods.

Moreover, we generalize Tatarski results, our formalism allowing us to relaxe some of the assumptions on the turbulent medium and to take into account vertically non-homogeneous correlation functions.

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